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Double-Change Circular Covering Designs: Constructions And Applications

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DOUBLE-CHANGE CIRCULAR COVERING DESIGNS: CONSTRUCTIONS AND APPLICATIONS

by

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TITLE: DOUBLE-CHANGE CIRCULAR COVERING DESIGNS: CONSTRUCTIONS AND APPLICATIONS

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A *double-change circular covering design (dcccd)* is an ordered set of blocks with block size k is an ordered collection of b block, $\mathcal{B} = \{B_1, \cdots, B_b\}$, each an unordered subset of k distinct elements from $[v]$, which obey: (1) each block differs from the previous block by two elements, as does the last from the first, and, (2) every unordered pair of $[v]$ appears in at least one block. The first object is to minimize b for a fixed v when $k = 3$ and arrange them in a circular manner. And the second object is to determine whether the covering designs are *economical* or *tight*.

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A special thanks also to C. C. Lindner and C. A. Rodger [3], from whose book many of the definitions and Bose and Skolem constructions used in this thesis have been quoted.

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INTRODUCTION

The theory of Statictical and Combinatoral Designs was developed last century by statisticians and mathematicians. It is now an important branch of combinatorics with researchers, both industrial and academic worldwide.

Designs for experiments are used worldwide by agricultural scientists when testing new fertilizers, pharmaceutical companies in testing new drugs, and sport organizations for arranging game schedules. Any advance in the theory of Designs aids these people.

In Chapter 1, general definitions and properties of double-change covering designs are introduced. And definitions of Latin square and quasigroups are given is the first chapter. We will also give useful theorems that are used in next two chapters.

Chapter 2 deals with the Steiner Triple systems. By constructing Steiner Triple systems we find that tight and economical circular double-change covering designs exists for $v \equiv 1$ or $3 \pmod 6$.

In chapter 3 we will give constructions for circular double-change covering designs for $v \equiv 0, 2, 4, 5 \pmod 6$ when $k = 3$ and discuss whether they are economical or tight.

CHAPTER 1

DOUBLE CHANGE COVERING DESIGNS

1.1 INTRODUCTION TO COMBINATORIAL COVERING DESIGNS

Combinatorial Design Theory is the study of arranging elements of a finite set into subsets so that certain specified properties are satisfied. And if each pair of elements from the set is covered by at least one subset then the design is called a *covering design*.

For example, suppose it is required to select 3-sets from the seven objects {1,2,3,4,5,6,7} in such a way that each object occurs in three of the 3-sets and every intersection of two 3-sets has precisely one member. The solution to this problem-the way of selecting the 3-sets-is a combinatorial design. One possible solution is {123,145,167,246,257,347,356} where 123 represents {1,2,3} and so on [6].

Many of the fundamental questions which combinatorial design theory concerns are existence problems: Does a design of a specified type exist? Modern design theory includes many existence results as well as nonexistence results.

Design theory is a field of combinatorics which makes use of tools from linear algebra, group theory, the theory of finite fields, number theory as well as combinatorics and with applications in areas such as statistics, computer science, biology, engineering and tournament scheduling.

We will use a specific type of covering design called a *block design* throughout this paper. Consider the following example: Suppose a company wants to test certain number, say v, types of cars to see which is the best? with minimum number of drivers, say b. It would be expensive and time consuming if each driver compared all v cars, so the company decides to have each driver compare a portion of cars. So the company forms subsets of v into *blocks*, say $B_1, B_2, \cdots, B_b$, to test for the b drivers. Thus a driver can make comparisons of each pair of cars in the given block; but in order to be efficient, we wish that each pair of cars is tested exactly once. hence each pair of cars must appear in exactly one block. Answers to questions such as this one will be revealed in this paper.

1.2 BASIC DEFINITIONS AND PROPERTIES

Definition 1.2.1. A design is a pair $(X, \mathcal{B})$ such that the following properties are satisfied:

- X is a set of elements called points.
- $\mathcal{B}$ is a collection of nonempty subsets of X called blocks.

Balanced incomplete block designs are probably the most-studied type of design. The study of balanced incomplete block designs was begun in the 1930s by Fisher and Yates [5].

Definition 1.2.2. Let v, k and λ be positive integers such that $v > k \geq 2$. A $(v, k, \lambda)-$balanced incomplete block design ($(v, k, \lambda)-$BIBD) is a design $(X, \mathcal{B})$ such that the following properties are satisfied:

1. $|X| = v$
2. each block contains exactly k points
3. every pair of distinct points is contained in exactly λ blocks.

Property 3 in the definition above is the "balance" property. A $BIBD$ is called an *incomplete block design* because $k < v$, and hence all its blocks are incomplete blocks.

Example 1.2.1. A $(7, 3, 1)-$ BIBD.

$X = \{1, 2, 3, 4, 5, 6, 7\}$ and

$\mathcal{B} = \{123, 145, 167, 246, 257, 347, 356\}$

This BIBD has a nice diagrammatic representation; see Figure 1.1. The blocks of the BIBD are the six lines and the circle in this diagram [1], [5].

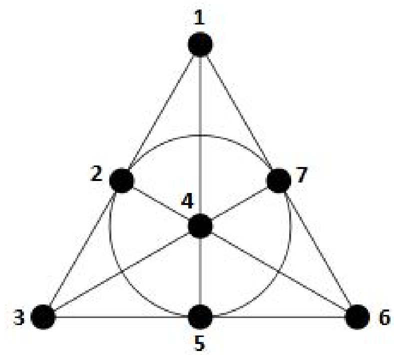

Figure 1.1. The Fano plane: A (7,3,1)-BIBD

Definition 1.2.3. A *double-change covering design(dccd)* based on the set $[v] = \{1, 2, \cdots, v\}$ with block size k is an ordered collection of b block, $\mathcal{B} = \{B_1, \cdots, B_b\}$, each an unordered subset of k distinct elements from $[v]$, which obey:

1. each block differs from the previous block by two elements, i.e., $|B_{i-1} \cap B_i| = k - 2$ for $i = 2, \cdots, b$;

2. every unordered pair $\{x, y\}$ of $[v]$, with $x \neq y$, appears in at least one block

We say the design is circular if the last block, B_b, differs from the first, B_1, by two elements, i.e., $|B_b \cap B_1| = k - 2$ and we say an element is *introduced* in a block if it one of the two new elements changed in the block. The term 'covering' connotes that all pairs of elements from X are covered within blocks [4].

For example, a *dccd* with $(v,k) = (7,3)$ is

1	2	3
1	4	6
1	5	7
3	5	4
2	5	6
3	7	6
2	7	4

We now state and prove the following general result of BIBDs.

Theorem 1.2.1. *In a $D = (v,k,\lambda)$ design based on S, let b be the number of blocks and let r be the repetition number(i.e. the number of times each element occurs). Then $\lambda(v-1) = r(k-1)$ and $bk = vr$.*

Proof : Choose any $x \in S$. Let it occur r_x times in D. So, the number of pairs (x,y) is $r_x(k-1)$ since there are $k-1$ such pairs per block and r_x such blocks. Now x occurs with $v-1$ other elements y in pair (x,y), each pair occurs λ times. So, the number of pairs containing x is also $\lambda(v-1)$. So, $r_x(k-1) = \lambda(v-1)$. So, $r_x = \frac{\lambda(v-1)}{k-1}$. Since r_x is uniquely determined by v, k and λ it is independent of the choice of x. So $r = r_x = \frac{\lambda(v-1)}{k-1}$ and hence$\lambda(v-1) = r(k-1)$.

To prove that $bk = vr$, note that each block has k elements and so the b blocks contain bk elements including repetitions. But each x occurs r times in the blocks. So we must have $bk = vr$.

For a fixed v and k, where $v \geq k+1$ and $k \geq 2$, we denote by $b_*(v,k)$ the smallest b for which there exists circular *dccd*.

Theorem 1.2.2. *For $v \geq 4$ and $k \geq 3$, the value of $b_*(v,k)$ satisfies*

$$b_*(v,k) \geq \left\lceil \frac{\binom{v}{2}}{2k-3} \right\rceil$$

Proof : A circular *dccd* must cover all $\binom{v}{2}$ pairs. Since two elements are introduced in each block, $2(k-2)+1$ pairs are covered per block. Thus,

$$b_*(v,k) * (2k-3) \geq \binom{v}{2}$$
$$b_*(v,k) \geq \left\lceil \frac{\binom{v}{2}}{2k-3} \right\rceil.$$

If $b_*(v,k) = \frac{\binom{v}{2}}{2k-3}$ then the design is called *tight* and if $b_*(v,k) = \left\lceil \frac{\binom{v}{2}}{2k-3} \right\rceil$ then the design is called *economical*.

1.3 LATIN SQUARES AND QUASIGROUPS

Definition 1.3.1. A Latin square of order n with entries from an $n-$set X is an $n \times n$ array L in which every cell contains an element of X such that every row and every column of L is a permutation of X.

Example 1.3.1. A Latin square of order 4.

1	2	3	4
4	1	3	2
3	4	1	2
2	3	4	1

Definition 1.3.2. A quasigroup of order n is a pair $(Q, \circ)$, where Q is a set of size n and "$\circ$" is a binary operation on Q such that for every pair of elements $a, b \in Q$, the equations $a \circ x = b$ and $y \circ a = b$ have unique solutions.

Example 1.3.2. A quasigroup of order 3.

$\circ$	1	2	3
1	1	2	3
2	3	1	2
3	2	3	1

Definition 1.3.3. A Latin square is said to be *idempotent* if cell (i, i) contains symbol i for $1 \leq i \leq n$. A Latin square is said to be *commutative* if cells (i, j) and (j, i) contain the same symbol for all $1 \leq i, j \leq n$.

Example 1.3.3. The following Latin square is both idempotent and commutative.

1	4	2	5	3
4	2	5	3	1
2	5	3	1	4
5	3	1	4	2
3	1	4	2	5

Definition 1.3.4. A Latin square (quasigroup) of order $2n$ is said to be *half-idempotent* if for $1 \leq i \leq n$ cells (i, i) and $(n + i, n + i)$ contains the symbol i.

Example 1.3.4. The following Latin square is both half-idempotent and commutative.

$\circ$	1	2	3	4
1	1	3	2	4
2	3	2	4	1
3	2	4	1	3
4	4	1	3	2

The building blocks we need for the Bose construction and for the Skolem construction are idempotent commutative quasigroup of order $2n+1$ and half-idempotent commutative quasigroup of order $2n$ respectively.

CHAPTER 2

THE STEINER TRIPLE SYSTEMS

2.1 INTRODUCTION TO STEINER TRIPLE SYSTEMS

Definition 2.1.1. A *Steiner Triple System* denoted by $STS(v)$ is a pair (S, T) consisting of a set S with v elements and a set T consisting of triples of S (called **blocks**) such that every pair of elements of S appear together in a unique triple of T.

In other words a $(v, 3, 1)$ design is a Steiner Triple System.

Example 2.1.1. Consider the complete graph K_v on v vertices . A decomposition of K_v into triangles(K_3's) with no common edges is equivalent to a Steiner Triple System.

Example 2.1.2. ***Kirkman's school girls problem (1850):*** *Fifteen young ladies in a school walk out three abreast for seven days in succession: it is required to arrange them daily so that no two shall walk twice abreast* [2], [6].
A solution to this problem is a $STS(15)$. If the girls are numbered from 1 to 15, the following is one solution.

Table 2.1. A solution of Kirkman's school girl problem

Mon	Tue	Wed	Thu	Fri	Sat	Sun
1,2,3	1,4,5	2,4,6	3,4,7	1,6,7	3,5,6	2,5,7
4,10,14	2,13,15	1,8,9	2,12,14	2,9,11	2,8,10	1,14,15
7,8,15	3,9,10	3,12,15	1,10,11	4,8,12	4,11,15	4,9,13
5,9,12	6,8,14	5,11,14	5,8,13	3,13,14	1,12,13	3,8,11
6,11,13	7,11,12	7,10,13	6,9,15	5,10,15	7,9,14	6,10,12

Steiner triple system can exist only for certain values of v. Note that in a $STS(v)$, $\lambda = 1$ and $k = 3$. Therefore by the Theorem 1.2.1, we have $v - 1 = 2r$ and $3b = vr$. This gives $b = \frac{v(v-1)}{6}$.

Theorem 2.1.1. *There exist an $STS(v)$ only if $v \equiv$1 or 3* $\pmod 6$.

Proof : Since $k = 3$ and $\lambda = 1$ we have $v = 2r + 1$; i.e., v is odd. Since $b = \frac{v(v-1)}{6}$ and b must be an integer we will have $v(v-1) \equiv 0 \pmod 6$. This is satisfied if and only if $v \equiv 0, 1, 3, 4 \pmod 6$. However since v is odd we have $v \equiv 1$ or $3 \pmod 6$.

2.2 CONSTRUCTION OF A CIRCULAR $DCCD$ FOR $V \equiv 3 \pmod 6$

The Bose Construction: Let $v = 6n + 3$ and let $(Q, \circ)$ be an idempotent commutative quasigroup of order $2n + 1$, where $Q = \{1, 2, 3, \cdots, 2n + 1\}$. Let $S = Q \times \{1, 2, 3\}$, and define T to contain the following two types of triples. [3]

Type 1: $\{(i, 1), (i, 2), (i, 3)\}$ for $1 \le i \le 2n + 1$

Type 2: $\{(i, 1), (j, 1), (i \circ j, 2)\}, \{(i, 2), (j, 2), (i \circ j, 3)\}, \{(i, 3), (j, 3), (i \circ j, 1)\}$

for $1 \le i < j \le 2n + 1$

Then (S, T) is a Steiner triple system of order $6n + 3$, $STS(6n + 3)$.

Example 2.2.1. Construct an $STS(9)$ using the Bose construction.

We will use the following idempotent commutative quasigroup of order 3.

$\circ$	1	2	3
1	1	3	2
2	3	2	1
3	2	1	3

Let $S = \{1,2,3\} \times \{1,2,3\}$. Then T contains the following 12 triples:

Type 1: {(1,1),(1,2),(1,3)},{(2,1),(2,2),(2,3)},{(3,1),(3,2),(3,3)}

Type 2:	$\{(1,1),(2,1),(3,2)\}$	$\{(1,1),(3,1),(2,2)\}$	$\{(2,1),(3,1),(1,2)\}$
	$\{(1,2),(2,2),(3,3)\}$	$\{(1,2),(3,2),(2,3)\}$	$\{(2,2),(3,2),(1,3)\}$
	$\{(1,3),(2,3),(3,1)\}$	$\{(1,3),(3,3),(2,1)\}$	$\{(2,3),(3,3),(1,1)\}$

2.2.1 Construction of a circular *dccd* for $v = 6n+3$ when $n \geq 1$

Let

- B_1 be the set of blocks of Type 2 of the form $\{(i,1),(j,1),(i \circ j,2)\}$ for $1 \leq i < j \leq 2n+1$
- B_2 be the set of blocks of Type 2 of the form $\{(i,2),(j,2),(i \circ j,3)\}$ for $1 \leq i < j \leq 2n+1$
- B_3 be the set of blocks of Type 2 of the form $\{(i,3),(j,3),(i \circ j,1)\}$ for $1 \leq i < j \leq 2n+1$

Arrangement of B_k for $k = 1, 2$ and 3: Since the third element of each block is uniquely determined by the construction if we arrange the first two elements as a single change design we will get blocks with a double change. Arrange the first two elements using following steps [7].

- Fix $i = 1$. Arrange the blocks by changing j from 2 to $2n+1$. So, the last block will be $\{(1,k),(2n+1,k),*\}$
- Increase i up to $2n$ while keeping j as $2n+1$ until $\{(2n,k),(2n+1,k),*\}$ is reached
- When $n = 1$ Stop the process since we have reached $\{(2,k),(3,k),*\}$ otherwise continue to the following steps.

- Fix $j = 2n$ and increase i from 2 to $2n-1$ until $\{(2n-1,k),(2n,k),*\}$ is reached
- Fix $j = 2n-1$ and increase i from 2 to $2n-2$ until $\{(2n-2,k),(2n-1,k),*\}$ is reached
- Repeat the process until $\{(2,k),(3,k),*\}$ is reached

Example 2.2.2. Let $v = 9$ then $n = 1$ and following is an idempotent commutative quasi-group of order 3 and a circular $dccd(9,3)$ with 12 blocks.

$\circ$	1	2	3
1	1	3	2
2	3	2	1
3	2	1	3

$(1,1),(2,1),(3,2)$	$(1,2),(2,2),(3,3)$	$(1,3),(2,3),(3,1)$
$(1,1),(3,1),(2,2)$	$(1,2),(3,2),(2,3)$	$(1,3),(3,3),(2,1)$
$(2,1),(3,1),(1,2)$	$(2,2),(3,2),(1,3)$	$(2,3),(3,3),(1,1)$
$(2,1),(2,2),(2,3)$	$(1,1),(1,2),(1,3)$	$(3,1),(3,2),(3,3)$

And if $n \geq 2$, let B_1' be the set of blocks obtained from B_1 by applying the following changes.

- Swap the first two blocks i.e. $\{(1,1),(2,1),*\}$ and $\{(1,1),(3,1),*\}$
- Insert the block $\{(2n+1,1),(2n+1,2),(2n+1,3)\}$ from the Type 1 blocks in between the two blocks $\{(1,1),(2n+1,1),*\}$ and $\{(2,1),(2n+1,1),*\}$
- For $4 \leq i \leq 2n$, insert the block $\{(i,1),(i,2),(i,3)\}$ from the Type 1 blocks in between the two blocks $\{(i,1),(i+1,1),*\}$ and $\{(2,1),(i,1),*\}$

Then use the following steps to construct a circular double change covering design for $n \geq 2$.

- **Step 1:** List the blocks of B_1'.

- **Step 2:** Insert the block $\{(2,1),(2,2),(2,3)\}$ from the Type 1 blocks. This can be done always since the last block of B_1' is $\{(2,1),(3,1),*\}$.

- **Step 3:** List the blocks of B_2. This can be done always since the first block of B_2 is $\{(1,2),(2,2),*\}$. Then swap the two blocks $\{(1,2),(3,2),(1 \circ 3,3)\}$ and $\{(2,2),(3,2),(2 \circ 3,3)\}$.

- **Step 4:** Insert the block $\{(1,1),(1,2),(1,3)\}$ from the Type 1 blocks. This can be done always since the last block of B_2 is $\{(1,2),(3,2),*\}$.

- **Step 5:** List the blocks of B_3. This can be done always since the first block of B_3 is $\{(1,3),(2,3),*\}$. Then insert the block $\{(3,1),(3,2),(3,3)\}$.

Note that in this construction we have

- $2n+1$ blocks of Type 1

- $n(2n+1)*3$ blocks of Type 2.

Thus, the number of blocks $b = 6n^2+5n+1$. And

$$\frac{v(v-1)}{6} = \frac{(6n+3)(6n+2)}{6} = 6n^2+5n+1.$$

Since $\lceil 6n^2+5n+1 \rceil = 6n^2+5n+1$, this design is both *economical* and *tight* by the **Theorem 1.2.2**.

Example 2.2.3. Let $v = 15$ then $n = 2$ and following is an idempotent commutative quasigroup of order 5 and a circular $dccd(15,3)$ with 35 blocks.

$\circ$	1	2	3	4	5
1	1	4	2	5	3
2	4	2	5	3	1
3	2	5	3	1	4
4	5	3	1	4	2
5	3	1	4	2	5

$(1,1),(3,1),(2,2)$	$(2,1),(2,2),(2,3)$	$(1,3),(2,3),(4,1)$
$(1,1),(2,1),(4,2)$	$(1,2),(2,2),(4,3)$	$(1,3),(3,3),(2,1)$
$(1,1),(4,1),(5,2)$	$(2,2),(3,2),(5,3)$	$(1,3),(4,3),(5,1)$
$(1,1),(5,1),(3,2)$	$(1,2),(4,2),(5,3)$	$(1,3),(5,3),(3,1)$
$(5,1),(5,2),(5,3)$	$(1,2),(5,2),(3,3)$	$(2,3),(5,3),(1,1)$
$(2,1),(5,1),(1,2)$	$(2,2),(5,2),(1,3)$	$(3,3),(5,3),(4,1)$
$(3,1),(5,1),(4,2)$	$(3,2),(5,2),(4,3)$	$(4,3),(5,3),(2,1)$
$(4,1),(5,1),(2,2)$	$(4,2),(5,2),(2,3)$	$(2,3),(4,3),(3,1)$
$(4,1),(4,2),(4,3)$	$(2,2),(4,2),(3,3)$	$(3,3),(4,3),(1,1)$
$(2,1),(4,1),(3,2)$	$(3,2),(4,2),(1,3)$	$(2,3),(3,3),(5,1)$
$(3,1),(4,1),(1,2)$	$(1,2),(3,2),(2,3)$	$(3,1),(3,2),(3,3)$
$(2,1),(3,1),(5,2)$	$(1,1),(1,2),(1,3)$	

Example 2.2.4. Let $v = 21$ then $n = 3$ and following is an idempotent commutative quasigroup of order 7 and a circular $dccd(21, 3)$ with 70 blocks.

$\circ$	1	2	3	4	5	6	7
1	1	5	2	6	3	7	4
2	5	2	6	3	7	4	1
3	2	6	3	7	4	1	5
4	6	3	7	4	1	5	2
5	3	7	4	1	5	2	6
6	7	4	1	5	2	6	3
7	4	1	5	2	6	3	7

$(1,1),(3,1),(5,2)$	$(2,1),(3,1),(6,2)$	$(1,3),(2,3),(5,1)$
$(1,1),(2,1),(2,2)$	$(2,1),(2,2),(2,3)$	$(1,3),(3,3),(2,1)$
$(1,1),(4,1),(6,2)$	$(1,2),(2,2),(5,3)$	$(1,3),(4,3),(6,1)$
$(1,1),(5,1),(3,2)$	$(2,2),(3,2),(6,3)$	$(1,3),(5,3),(3,1)$
$(1,1),(6,1),(7,2)$	$(1,2),(4,2),(6,3)$	$(1,3),(6,3),(7,1)$
$(1,1),(7,1),(4,2)$	$(1,2),(5,2),(3,3)$	$(1,3),(7,3),(4,1)$
$(7,1),(7,2),(7,3)$	$(1,2),(6,2),(7,3)$	$(2,3),(7,3),(1,1)$
$(2,1),(7,1),(1,2)$	$(1,2),(7,2),(4,3)$	$(3,3),(7,3),(5,1)$
$(3,1),(7,1),(5,2)$	$(2,2),(7,2),(1,3)$	$(4,3),(7,3),(2,1)$
$(4,1),(7,1),(2,2)$	$(3,2),(7,2),(5,3)$	$(5,3),(7,3),(6,1)$
$(5,1),(7,1),(6,2)$	$(4,2),(7,2),(2,3)$	$(6,3),(7,3),(3,1)$
$(6,1),(7,1),(3,2)$	$(5,2),(7,2),(6,3)$	$(2,3),(6,3),(4,1)$
$(6,1),(6,2),(6,3)$	$(6,2),(7,2),(3,3)$	$(3,3),(6,3),(1,1)$
$(2,1),(6,1),(4,2)$	$(2,2),(6,2),(4,3)$	$(4,3),(6,3),(5,1)$
$(3,1),(6,1),(1,2)$	$(3,2),(6,2),(1,3)$	$(5,3),(6,3),(2,1)$
$(4,1),(6,1),(5,2)$	$(4,2),(6,2),(5,3)$	$(2,3),(5,3),(7,1)$
$(5,1),(6,1),(2,2)$	$(5,2),(6,2),(2,3)$	$(3,3),(5,3),(4,1)$
$(5,1),(5,2),(5,3)$	$(2,2),(5,2),(7,3)$	$(4,3),(5,3),(1,1)$
$(2,1),(5,1),(7,2)$	$(3,2),(5,2),(4,3)$	$(2,3),(4,3),(3,1)$
$(3,1),(5,1),(4,2)$	$(4,2),(5,2),(1,3)$	$(3,3),(4,3),(7,1)$
$(4,1),(5,1),(1,2)$	$(2,2),(4,2),(3,3)$	$(2,3),(3,3),(6,1)$
$(4,1),(4,2),(4,3)$	$(3,2),(4,2),(7,3)$	$(3,1),(3,2),(3,3)$
$(2,1),(4,1),(3,2)$	$(1,2),(3,2),(2,3)$	
$(3,1),(4,1),(7,2)$	$(1,1),(1,2),(1,3)$	

2.3 CONSTRUCTION OF A CIRCULAR $DCCD$ FOR $V \equiv 1 \pmod 6$

The Skolem Construction: Let $v = 6n+1$ and let $(Q, \circ)$ be a half-idempotent commutative quasigroup of order $2n$, where $Q = \{1, 2, 3, \cdots, 2n\}$. Let $S = \{\infty\} \cup (Q \times \{1, 2, 3\})$, and define T to contain the following three types of triples.[3]

Type 1: $\{(i, 1), (i, 2), (i, 3)\}$ for $1 \leq i \leq n$

Type 2: $\{\infty, (n+i, 1), (i, 2)\}, \{\infty, (n+i, 2), (i, 3)\}, \{\infty, (n+i, 3), (i, 1)\}$ for $1 \leq i \leq n$

Type 3: $\{(i, 1), (j, 1), (i \circ j, 2)\}, \{(i, 2), (j, 2), (i \circ j, 3)\}, \{(i, 3), (j, 3), (i \circ j, 1)\}$

for $1 \leq i < j \leq 2n$

Then (S, T) is a Steiner triple system of order $6n+1$.

Example 2.3.1. Construct an $STS(7)$ using the Skolem construction.

We will use the following half-idempotent commutative quasigroup of order 2.

$\circ$	1	2
1	1	2
2	2	1

Let $S = \{\infty\} \cup (\{1, 2\} \times \{1, 2, 3\})$. Then T contains the following 7 triples:

Type 1: {(1,1),(1,2),(1,3)}

Type 2: $\{\infty, (2, 1), (1, 2)\}, \{\infty, (2, 2), (1, 3)\}, \{\infty, (2, 3), (1, 1)\}$

Type 3: {(1,1),(2,1),(2,2)}, {(1,2),(2,2),(2,3)}, {(1,3),(2,3),(2,1)}

Then the following is a circular $dccd(7, 3)$ with 7 blocks. i.e. the $6n + 1$ construction when $n = 1$.

(1,1)	(1,2)	(1,3)
∞	(2,1)	(1,2)
∞	(2,2)	(1,3)
∞	(2,3)	(1,1)
(1,1)	(2,1)	(2,2)
(1,2)	(2,2)	(2,3)
(1,3)	(2,3)	(2,1)

2.3.1 Construction of a circular *dccd* for $v = 6n + 1$ when $n \geq 2$

Let

- B_1 be the set of blocks of Type 3 of the form $\{(i, 1), (j, 1), (i \circ j, 2)\}$ for $1 \leq i < j \leq 2n$
- B_2 be the set of blocks of Type 3 of the form $\{(i, 2), (j, 2), (i \circ j, 3)\}$ for $1 \leq i < j \leq 2n$
- B_3 be the set of blocks of Type 3 of the form $\{(i, 3), (j, 3), (i \circ j, 1)\}$ for $1 \leq i < j \leq 2n$

Arrangement of B_k for $k = 1, 2$ and 3: Since the third element of each block is uniquely determined by the construction if we arrange the first two elements as a single change design we will get blocks with a double change. Arrange the first two elements using the following steps.

- Fix $i = 1$. Arrange the blocks by changing j from 2 to $2n$. So, the last block will be $\{(1, k), (2n, k), *\}$

- Increase i up to $2n-1$ while keeping j as $2n$ until $\{(2n-1,k),(2n,k),*\}$ is reached

- Fix $j = 2n-1$ and increase i from 2 to $2n-2$ until $\{(2n-2,k),(2n-1,k),*\}$ is reached. When $n = 2$ stop the process since we have reached $\{(2,k),(3,k),*\}$ otherwise continue to the following steps.

- Fix $j = 2n-2$ and increase i from 2 to $2n-3$ until $\{(2n-3,k),(2n-2,k),*\}$ is reached

- Repeat the process until $\{(2,k),(3,k),*\}$ is reached

Example 2.3.2. Let $v = 13$ then $n = 2$ and following is a half-idempotent commutative quasigroup of order 4 and a circular $dccd(13,3)$ with 26 blocks.

$\circ$	1	2	3	4
1	1	3	2	4
2	3	2	4	1
3	2	4	1	3
4	4	1	3	2

$(1,1),(1,2),(1,3)$	$(1,1),(4,1),(4,2)$	$(3,2),(4,2),(3,3)$
$\infty,(3,1),(1,2)$	$(2,1),(4,1),(1,2)$	$(2,2),(3,2),(4,3)$
$\infty,(4,1),(2,2)$	$(3,1),(4,1),(3,2)$	$(2,3),(4,3),(1,1)$
$\infty,(3,2),(1,3)$	$(2,1),(3,1),(4,2)$	$(1,3),(2,3),(3,1)$
$\infty,(4,2),(2,3)$	$(2,1),(2,2),(2,3)$	$(1,3),(3,3),(2,1)$
$\infty,(3,3),(1,1)$	$(1,2),(2,2),(3,3)$	$(3,3),(4,3),(3,1)$
$\infty,(4,3),(2,1)$	$(1,2),(3,2),(2,3)$	$(2,3),(3,3),(4,1)$
$(1,1),(2,1),(3,2)$	$(1,2),(4,2),(4,3)$	$(1,3),(4,3),(4,1)$
$(1,1),(3,1),(2,2)$	$(2,2),(4,2),(1,3)$	

And if $n \geq 3$, let $B_1^{'}$ be the set of blocks obtained from B_1 by applying the following changes.

- Take the block $\{1,1),(n,1),(1 \circ n,2)\}$ to the top of B_1. If $n = 3$ this is the only change.

- If $n > 3$, for the $n-3$ blocks of Type 1 from the block $\{(4,1),(4,2),(4,3)\}$ to the block $\{(n,1),(n,2),(n,3)\}$ insert the block $\{(t,1),(t,2),(t,3)\}$ in between the two blocks $\{(t,1),(t+1,1),*\}$ and $\{(2,1),(t,1),*\}$ where $4 \leq t \leq n$.

Then use the following steps to construct a circular double change covering design for $n \geq 3$.

- **Step 1:** List the block {(1,1),(1,2),(1,3)} of Type 1.

- **Step 2:** List the Type 2 blocks. This can be done since the first block of Type 2 is $\{\infty,(n+1,1),(1,2)\}$.

- **Step 3:** List the blocks of $B_1^{'}$. Since the last block of Type 2 is $\{\infty,(2n,3),(n,1)\}$ and the first block of $B_1^{'}$ is $\{(1,1),(n,1),(1 \circ n,2)\}$.

- **Step 4:** Insert the block $\{(2,1),(2,2),(2,3)\}$ from Type 1 blocks. This can be done since the last block of B_1' is $\{(2,1),(3,1),(2 \circ 3, 2)\}$.

- **Step 5:** List the blocks of B_2. This can be done since the first block of B_2 is $\{(1,2),(2,2),(1 \circ 2, 3)\}$.

- **Step 6:** Insert the block $\{(3,1),(3,2),(3,3)\}$ from the Type 1 blocks. This can be done since the last block of B_2 is $\{(2,2),(3,2),(2 \circ 3, 3)\}$.

- **Step 7:** List the blocks of B_3. Then take the first block of B_3 i.e. the block $\{(1,3),(2,3),*\}$ to the bottom of B_3.

Note that in this construction we have

- n blocks of Type 1
- $3n$ blocks of Type 2
- $n(2n-1) * 3$ blocks of Type 3.

Thus, the number of blocks $b = 6n^2 + n$. And

$$\frac{v(v-1)}{6} = \frac{(6n+1)(6n)}{6} = 6n^2 + n.$$

Since $\lceil 6n^2 + n \rceil = 6n^2 + n$, this design is both *economical* and *tight* by the **Theorem 1.2.2**.

Example 2.3.3. Let $v = 19$ then $n = 3$ and following is a half-idempotent commutative quasigroup of order 6 and a circular $dccd(19,3)$ with 57 blocks.

$\circ$	1	2	3	4	5	6
1	1	4	2	5	3	6
2	4	2	5	3	6	1
3	2	5	3	6	1	4
4	5	3	6	1	4	2
5	3	6	1	4	2	5
6	6	1	4	2	5	3

$(1,1),(1,2),(1,3)$	$(2,1),(5,1),(6,2)$	$(2,2),(4,2),(3,3)$
$\infty,(4,1),(1,2)$	$(3,1),(5,1),(1,2)$	$(3,2),(4,2),(6,3)$
$\infty,(5,1),(2,2)$	$(4,1),(5,1),(4,2)$	$(2,2),(3,2),(5,3)$
$\infty,(6,1),(3,2)$	$(2,1),(4,1),(3,2)$	$(3,1),(3,2),(3,3)$
$\infty,(4,2),(1,3)$	$(3,1),(4,1),(6,2)$	$(1,3),(3,3),(2,1)$
$\infty,(5,2),(2,3)$	$(2,1),(3,1),(5,2)$	$(1,3),(4,3),(5,1)$
$\infty,(6,2),(3,3)$	$(2,1),(2,2),(2,3)$	$(1,3),(5,3),(3,1)$
$\infty,(4,3),(1,1)$	$(1,2),(2,2),(4,3)$	$(1,3),(6,3),(6,1)$
$\infty,(5,3),(2,1)$	$(1,2),(3,2),(2,3)$	$(2,3),(6,3),(1,1)$
$\infty,(6,3),(3,1)$	$(1,2),(4,2),(5,3)$	$(3,3),(6,3),(4,1)$
$(1,1),(3,1),(2,2)$	$(1,2),(5,2),(3,3)$	$(4,3),(6,3),(2,1)$
$(1,1),(2,1),(4,2)$	$(1,2),(6,2),(6,3)$	$(5,3),(6,3),(5,1)$
$(1,1),(4,1),(5,2)$	$(2,2),(6,2),(1,3)$	$(2,3),(5,3),(6,1)$
$(1,1),(5,1),(3,2)$	$(3,2),(6,2),(4,3)$	$(3,3),(5,3),(1,1)$
$(1,1),(6,1),(6,2)$	$(4,2),(6,2),(2,3)$	$(4,3),(5,3),(4,1)$
$(2,1),(6,1),(1,2)$	$(5,2),(6,2),(5,3)$	$(2,3),(4,3),(3,1)$
$(3,1),(6,1),(4,2)$	$(2,2),(5,2),(6,3)$	$(3,3),(4,3),(6,1)$
$(4,1),(6,1),(2,2)$	$(3,2),(5,2),(1,3)$	$(2,3),(3,3),(5,1)$
$(5,1),(6,1),(5,2)$	$(4,2),(5,2),(4,3)$	$(1,3),(2,3),(4,1)$

CHAPTER 3

MINIMUM COVERING DESIGNS

3.1 THE GENERAL PROBLEM

As we know *a* $STS(v)$ *exists if and only if* $v \equiv 1$ *or* $3 \pmod 6$*,* a natural question to ask is how "close" can we come to constructing a triple system for $v \equiv 0, 2, 4$ or $5 \pmod 6$. In order to attack this problem we must first define what we mean by "close".

Definition 3.1.1. A *covering* of the complete graph K_v with triangles is a triple (S, T, P), where S is the vertex set of K_v, P is a subset of the edge set of λK_v based on S (λK_v is the graph in which each pair of vertices is joined by λ edges), and T is a collection of triangles which partitions the union of P and the edge set of K_v. The collection of edges P is called the *padding* and the number v the *order* of the covering (S, T, P).

If $|P|$ is as small as possible the covering (S, T, P) is called a *minimum covering with triangles* (MCT), or more simply a *minimum covering* of order v.

Example 3.1.1. Let

$$\begin{aligned} S &= \{1, 2, 3, 4, 5\} \\ T &= \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 5\}, \{3, 4, 5\}\} \\ P &= \{\{1, 2\}, \{1, 2\}\} \end{aligned}$$

then (S,T,P) is a minimum covering of order 5.

The padding of a minimum covering is determined by its order. In particular, if (S, T, P) is a MCT of order v, then the padding is [3]

(i) a 1-factor if $v \equiv 0 \pmod 6$

(ii) a tripole if $v \equiv 2$ or $4 \pmod 6$

(iii) a double edge if $v \equiv 5 \pmod 6$

(iv) the empty set if $v \equiv 1$ or $3 \pmod 6$

Definition 3.1.2. A *pairwise balanced design* (PBD) is an ordered pair (S, B), where S is a finite set of symbols, and B is a collection of subsets of S called *blocks*, such that each pair of distinct elements of S occurs together in exactly one block of B and $|S|$ is called the *order* of the PBD.

So a Steiner triple system is a minimum covering with padding $P = \emptyset$ and pairwise balanced design in which each block has size 3.

Example 3.1.2. Let $S = \{1, 2, 3, \cdots, 11\}$ and B contains the following blocks, then (S, B) is a PBD of order 11:

{1,2,3,4,5}	{2,7,8}	{3,7,9}	{4,8,10}
{1,6,7}	{2,9,10}	{3,8,11}	{5,6,8}
{1,8,9}	{2,6,11}	{4,6,9}	{5,7,10}
{1,10,11}	{3,6,10}	{4,7,11}	{5,9,11}

3.2 CONSTRUCTION OF CIRCULAR $DCCD$ FOR $V \equiv 5 \pmod 6$

We need to construct a $PBD(S, B)$ of order v with exactly one block of size 5 and the rest having size 3, for all $v \equiv 5 \pmod 6$.

The $6n + 5$ construction: Let $Q = \{1, 2, \cdots, 2n + 1\}$ and let α be the permutation $(1)(2\ 3 \cdots\ 2n + 1)$. Let $(Q, \circ)$ be an idempotent commutative quasigroup of order $2n + 1$. Let $S = \{\infty_1, \infty_2\} \cup (\{1, 2, \cdots, 2n + 1\} \times \{1, 2, 3\})$ and let B contain the following blocks:

Type 1: $\{\infty_1, \infty_2, (1,1), (1,2), (1,3)\}$

Type 2: $\{\infty_1, (2i,1), (2i,2)\}, \{\infty_1, (2i,3), (\alpha(2i),1)\}, \{\infty_1, (\alpha(2i),2), (\alpha(2i),3)\},$

$\{\infty_2, (2i,2), (2i,3)\}, \{\infty_2, (\alpha(2i),1), (\alpha(2i),2)\}, \{\infty_2, (\alpha(2i),1), (\alpha^{-1}(2i),3)\}$

for $1 \leq i \leq n$

Type 3: $\{(i,1), (j,1), (i \circ j, 2)\}, \{(i,2), (j,2), (i \circ j, 3)\}, \{(i,3), (j,3), (\alpha(i \circ j), 1)\}$

for $1 \leq i < j \leq 2n+1$.

Then (S, B) is a $PBD(6n+5)$ with exactly one block of size 5 and rest of size 3.

The Double Edge Covering Construction: Let $v \equiv 5 \pmod 6$ and let (S, B) be a PBD of order n with one block $\{a, b, c, d, e\}$ of size 5 (i.e the Type 1 block) and the remaining blocks of size 3. Denote by T the collection of blocks of size 3, and let $T^* = \{\{a,b,c\}, \{a,b,d\}, \{a,b,e\}, \{c,d,e\}\}$. Then $(S, T \cup T^*, P)$ is a minimum covering of order v, where $P = \{\{a,b\}, \{a,b\}\}$.

Example 3.2.1. Let $v = 5$, then $n = 0$ and B contains only the block {1,2,3,4,5} of Type 1. So $T = \emptyset$, $T^* = \{\{1,2,3\}, \{1,2,4\}, \{1,2,5\}, \{3,4,5\}\}$ and $P = \{\{1,2\}, \{1,2\}\}$. Then the design

1	2	3
1	2	4
1	2	5
3	4	5

is a covering design with 4 blocks, and

$$b = \left\lceil \frac{\binom{5}{2}}{2*3-3} \right\rceil = \left\lceil \frac{10}{3} \right\rceil = 4$$

So by the **Theorem 1.2.2** it is *economical* but not *tight*. But there is no *dccd* for $n = 0$.

3.2.1 Construction of a circular *dccd* for $v = 6n + 5$ when $n \geq 1$

Let

- B_1 be the set of blocks of Type 3 of the form $\{(i, 1), (j, 1), (i \circ j, 2)\}$ for $1 \leq i < j \leq 2n + 1$
- B_2 be the set of blocks of Type 3 of the form $\{(i, 2), (j, 2), (i \circ j, 3)\}$ for $1 \leq i < j \leq 2n + 1$
- B_3 be the set of blocks of Type 3 of the form $\{(i, 3), (j, 3), (\alpha(i \circ j), 1)\}$ for $1 \leq i < j \leq 2n + 1$
- B_{∞_1} be the set of blocks of Type 2 which has ∞_1 as the first element
- B_{∞_2} be the set of blocks of Type 2 which has ∞_2 as the first element

And there are four blocks of size 3 from the Type 1 block :

$$\{\infty_1, \infty_2, (1, 1)\}$$
$$\{\infty_1, \infty_2, (1, 2)\}$$
$$\{\infty_1, \infty_2, (1, 3)\}$$
$$\{(1, 1), (1, 2), (1, 3)\}$$

Arrangement of B_k for $k = 1, 2, 3$: Since the third element of each block is uniquely determined by the construction if we arrange the first two elements as a single change design we will get blocks with a double change. Arrange the first two elements using following steps.

- Fix $i = 1$. Arrange the blocks by changing j from 2 to $2n + 1$. So, the last block will be $\{(1, k), (2n + 1, k), *\}$
- Increase i up to $2n$ while keeping j as $2n + 1$ until $\{(2n, k), (2n + 1, k), *\}$ is reached.
- Fix $j = 2n$ and increase i from 2 to $2n - 1$ until $\{(2n - 1, k), (2n, k), *\}$ is reached.

- Fix $j = 2n-1$ and increase i from 2 to $2n-2$ until $\{(2n-2,k),(2n-1,k),*\}$ is reached.

- Repeat the process until $\{(2,k),(3,k),*\}$ is reached.

Arrangement of B_{∞_1}: For $1 \le i \le n$ write down the following blocks.

$$\{\infty_1,(2i,1),(2i,2)\}$$
$$\{\infty_1,(2i,3),(\alpha(2i),1)\}$$
$$\{\infty_1,(\alpha(2i),2),(\alpha(2i),3)\}$$

Arrangement of B_{∞_2}: For $1 \le i \le n$ write down the following blocks.

$$\{\infty_2,(2i,2),(2i,3)\}$$
$$\{\infty_2,(\alpha(2i),1),(\alpha(2i),2)\}$$
$$\{\infty_2,(2i,1),(\alpha^{-1}(2i),3)\}$$

Then use the following steps to construct a circular double change covering design.

- **Step 1:** Start with the $\{\infty_1,\infty_2,(1,1)\}$ block

- **Step 2:** Then, list the blocks from B_1. This can be done since the first block of B_1 is $\{(1,1),(2,1),(1\circ 2,2)\}$ and only common element is $(1,1)$

- **Step 3:** Next list the blocks from B_{∞_1}. This can always be done since the last block of B_1 is $\{(2,1),(3,1),(2\circ 3,2)\}$ and the first block of B_{∞_1} is $\{\infty_1,(2,1),(2,2)\}$. Further, $2\circ 3 = 1$ for $n=1$ and for $n \ge 2$, $2\circ 3 = 5$ and we interchange 5 with $n+3$ in the construction of idempotent commutative quassigroup. So, $(2\circ 3,2)$ never equals to $(2,2)$.

- **Step 4:** List the $\{\infty_1,\infty_2,(1,2)\}$ block. Since $\alpha(1)=1$, $(\alpha(2n),2)\neq(1,2)$ and hence the only common element is ∞_1

- **Step 5:** Now list the blocks from B_2. Since the first block of B_2 is $\{(1,2),(2,2),(1 \circ 2,3)\}$ the only common element is $(1,2)$.

- **Step 6:** Then list the blocks of B_{∞_2}. This can always be done from the same reasoning as in Step 3.

- **Step 7:** Next list the $\{\infty_1, \infty_2, (1,3)\}$ block. Since $\alpha(1) = 1$, $(\alpha^{-1}(2n), 3) \neq (1,3)$ and hence the only common element is ∞_2.

- **Step 8:** Now list the blocks from B_3. Since the first block of B_3 is $\{(1,3),(2,3),(1 \circ 2,1)\}$ the only common element is $(1,3)$.

- **Step 9:** Then swap the two blocks $\{(1,3),(3,3),(1 \circ 3,1)\}$ and $\{(2,3),(3,3),(2 \circ 3,1)\}$ in B_3. This can always be done since $(\alpha(2 \circ 3),1) = (\alpha(1 \circ 4),1)$ and $(\alpha(1 \circ 2),1) \neq (\alpha(2 \circ 3),1)$.

- **Step 10:** Finaly, write down the $\{(1,1),(1,2),(1,3)\}$ block. This can be done since $\alpha(1) = 1$ we have $(\alpha(1 \circ 3),1) \neq (1,1)$.

Then we will have a circular double change covering design.

Note that in this construction we have

- 4 blocks of Type 1
- $6n$ blocks of Type 2, since there are 6 blocks for each i and $1 \leq i \leq n$
- $n(2n+1) * 3$ blocks of Type 3.

Thus, the number of blocks $b = 6n^2 + 9n + 4$.

And
$$\frac{v(v-1)}{6} = \frac{(6n+5)(6n+4)}{6} = \frac{18n^2+27n+10}{3} = 6n^2 + 9n + \frac{10}{3}.$$
Since $\lceil 6n^2 + 9n + \frac{10}{3} \rceil = 6n^2 + 9n + 4$, this design is *economical* but not *tight* because the pair $\{\infty_1, \infty_2\}$ is repeated.

Example 3.2.2. Let $v = 11$ then $n = 1$ and following is an idempotent commutative quasigroup of order 3 used to construct the circular $dccd(11, 3)$ with 19 blocks below.

$\circ$	1	2	3
1	1	3	2
2	3	2	1
3	2	1	3

$\infty_1, \infty_2, (1,1)$	$\infty_1, \infty_2, (1,2)$	$\infty_1, \infty_2, (1,3)$
$(1,1), (2,1), (3,2)$	$(1,2), (2,2), (3,3)$	$(1,3), (2,3), (2,1)$
$(1,1), (3,1), (2,2)$	$(1,2), (3,2), (2,3)$	$(2,3), (3,3), (1,1)$
$(2,1), (3,1), (1,2)$	$(2,2), (3,2), (1,3)$	$(1,3), (3,3), (3,1)$
$\infty_1, (2,1), (2,2)$	$\infty_2, (2,2), (2,3)$	$(1,1), (1,2), (1,3)$
$\infty_1, (2,3), (3,1)$	$\infty_2, (3,1), (3,2)$	
$\infty_1, (3,2), (3,3)$	$\infty_2, (2,1), (3,3)$	

Example 3.2.3. Let $v = 17$ then $n = 2$ and following is an idempotent commutative quasigroup of order 5 used to construct the circular $dccd(17, 3)$ with 46 blocks below.

$\circ$	1	2	3	4	5
1	1	4	2	5	3
2	4	2	5	3	1
3	2	5	3	1	4
4	5	3	1	4	2
5	3	1	4	2	5

$\infty_1, \infty_2, (1,1)$	$\infty_1, \infty_2, (1,2)$	$\infty_1, \infty_2, (1,3)$
$(1,1), (2,1), (4,2)$	$(1,2), (2,2), (4,3)$	$(1,3), (2,3), (5,1)$
$(1,1), (3,1), (2,2)$	$(1,2), (3,2), (2,3)$	$(2,3), (3,3), (2,1)$
$(1,1), (4,1), (5,2)$	$(1,2), (4,2), (5,3)$	$(1,3), (4,3), (2,1)$
$(1,1), (5,1), (3,2)$	$(1,2), (5,2), (3,3)$	$(1,3), (5,3), (4,1)$
$(2,1), (5,1), (1,2)$	$(2,2), (5,2), (1,3)$	$(2,3), (5,3), (1,1)$
$(3,1), (5,1), (4,2)$	$(3,2), (5,2), (4,3)$	$(3,3), (5,3), (5,1)$
$(4,1), (5,1), (2,2)$	$(4,2), (5,2), (2,3)$	$(4,3), (5,3), (3,1)$
$(2,1), (4,1), (3,2)$	$(2,2), (4,2), (3,3)$	$(2,3), (4,3), (4,1)$
$(3,1), (4,1), (1,2)$	$(3,2), (4,2), (1,3)$	$(3,3), (4,3), (1,1)$
$(2,1), (3,1), (5,2)$	$(2,2), (3,2), (5,3)$	$(1,3), (3,3), (3,1)$
$\infty_1, (2,1), (2,2)$	$\infty_2, (2,2), (2,3)$	$(1,1), (1,2), (1,3)$
$\infty_1, (2,3), (3,1)$	$\infty_2, (3,1), (3,2)$	
$\infty_1, (3,2), (3,3)$	$\infty_2, (2,1), (3,3)$	
$\infty_1, (4,1), (4,2)$	$\infty_2, (4,2), (4,3)$	
$\infty_1, (4,3), (5,1)$	$\infty_2, (5,1), (5,2)$	
$\infty_1, (5,2), (5,3)$	$\infty_2, (4,1), (3,3)$	

3.3 CONSTRUCTION OF A CIRCULAR $DCCD$ FOR $V \equiv 0 \pmod 6$

Consider $v = 6(n+1)$, where $n = 0, 1, 2, \cdots$

The 1-factor Covering Construction: Let $v \equiv 0 \pmod 6$ and let (X, B) be a PBD of order $v - 1 \equiv 5 \pmod 6$ with one Type 1 block $\{a, b, c, d, e\}$ of size 5 and the remaining blocks of size 3. Denote by T the collection of blocks of size 3. Let $S = \{\infty\} \cup X$ and let $\pi = \{\{x_1, y_1\}, \{x_2, y_2\}, \cdots, \{x_t, y_t\}\}$ be any partition of $X \setminus \{a, b, c, d, e\}$. Let $\pi(\infty) = \{\{\infty, x_1, y_1\}, \{\infty, x_2, y_2\}, \cdots, \{\infty, x_t, y_t\}\}$ and $F(\infty) = \{\{\infty, a, e\}, \{\infty, b, e\}, \{\infty, c, d\}, \{a, b, c\}, \{a, b, d\}, \{c, d, e\}\}$. Then (S, T^*, P) is a minimum covering of order v, where $T^* = T \cup \pi(\infty) \cup F(\infty)$, and $P = \pi \cup \{\{a, b\}, \{c, d\}, \{e, \infty\}\}$. (Note that $F(\infty)$ is a MCT of order 6.)

Example 3.3.1. Consider the PBD of order 11 in **Example 3.1.2** with $\{a, b, c, d, e\} = \{1, 2, 3, 4, 5\}$. Then

$T = \{\{1, 6, 7\}, \{1, 8, 9\}, \{1, 10, 11\}, \{2, 7, 8\}, \{2, 9, 10\}, \{2, 6, 11\}, \{3, 6, 10\}, \{3, 7, 9\},$
$\{3, 8, 11\}, \{4, 6, 9\}, \{4, 7, 11\}, \{4, 8, 10\}, \{5, 6, 8\}, \{5, 7, 10\}, \{5, 9, 11\}\}.$

Let $\pi = \{\{6, 7\}, \{8, 9\}, \{10, 11\}\}$ then

$\pi(\infty) = \{\{\infty, 6, 7\}, \{\infty, 8, 9\}, \{\infty, 10, 11\}\}$ and

$F(\infty) = \{\{\infty, 1, 5\}, \{\infty, 2, 5\}, \{\infty, 3, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{3, 4, 5\}\}.$

Then (S,T^*,P) is a minimum covering of order 12, where $T^* = T \cup \pi(\infty) \cup F(\infty)$ and $P = \pi \cup \{\{1, 2\}, \{3, 4\}, \{5, \infty\}\}$

3.3.1 Construction of a circular $dccd$ for $v = 6(n+1)$ when $n = 0$

Consider $S = \{1, 2, 3, 4, 5, 6\}$ and let $X = \{1, 2, 3, 4, 5\}$. Since $v = 6$, (X, B) is a PBD of order 5 with one block of size 5. So, $T = \emptyset$ and $\pi = \emptyset$ and therefore $\pi(\infty) = \emptyset$. And $F(\infty) = F(6) = \{\{1, 5, 6\}, \{2, 5, 6\}, \{3, 4, 6\}, \{1, 2, 3\}, \{1, 2, 4\}, \{3, 4, 5\}\}$ and $P = \{\{1, 2\}, \{3, 4\}, \{5, 6\}\}$. Then the design

$$
\begin{array}{ccc}
1 & 2 & 3 \\
1 & 5 & 6 \\
3 & 4 & 6 \\
1 & 2 & 4 \\
3 & 4 & 5 \\
2 & 5 & 6
\end{array}
$$

is a circular *dccd* with 6 blocks. Since

$$\left\lceil \frac{\binom{6}{2}}{2 * 3 - 3} \right\rceil = \left\lceil \frac{15}{3} \right\rceil = 5 < 6$$

this design is neither *economical* nor *tight* by the **Theorem 1.2.2**.

3.3.2 Construction of a circular *dccd* for $v = 6(n+1)$ when $n \geq 1$

Consider the $6n+5$ construction with the Type 1 block $\{\infty_1, \infty_2, (1,1), (1,2), (1,3)\}$ of size 5 and Type 2 and Type 3 blocks of size 3. Then $F(\infty)$ contains the following blocks:

$\{\infty, \infty_1, (1,3)\}$

$\{\infty, \infty_2, (1,3)\}$

$\{\infty, (1,1), (1,2)\}$

$\{\infty_1, \infty_2, (1,1)\}$

$\{\infty_1, \infty_2, (1,2)\}$

$\{(1,1), (1,2), (1,3)\}$

Since π partitions the set $X \setminus \{\infty_1, \infty_2, (1,1), (1,2), (1,3)\}$, blocks of $\pi(\infty)$ form a double change design with ∞ as the common element. Then following is the procedure to construct a circular double change covering design.

- **Step 1:** Start with the $\{\infty_1, \infty_2, (1,1)\}$ block

- **Step 2:** Then list the blocks from B_1. This can be done since the first block of B_1 is $\{(1,1),(2,1),(1\circ 2,2)\}$ and the only common element is $(1,1)$.

- **Step 3:** Next list the blocks from B_{∞_1}. This can always be done since the last block of B_1 is $\{(2,1),(3,1),(2\circ 3,2)\}$ and the first block of B_{∞_1} is $\{\infty_1,(2,1),(2,2)\}$. Further, $2\circ 3 = 1$ for $n=1$ and for $n \geq 2$, $2\circ 3 = 5$ and we interchange 5 with $n+3$ in the construction of idempotent commutative quasigroup. So, $(2\circ 3, 2) \neq (2,2)$.

- **Step 4:** List the $\{\infty_1, \infty_2, (1,2)\}$ block. Since $\alpha(1) = 1$, $(\alpha(2n), 2) \neq (1,2)$ and hence the only common element is ∞_1.

- **Step 5:** Now list the blocks from B_2. Since the first block of B_2 is $\{(1,2),(2,2),(1\circ 2,3)\}$ the only common element is $(1,2)$.

- **Step 6:** Then list the blocks of B_{∞_2}. This can always be done by the same reasoning as in Step 3.

- **Step 7:** Next list the $\{\infty, \infty_2, (1,3)\}$ block. Since $\alpha(1) = 1$, $(\alpha^{-1}(2n), 3) \neq (1,3)$ and hence the only common element is ∞_2.

- **Step 8:** List the $\{\infty, (1,1), (1,2)\}$ block.

- **Step 9:** Next list the blocks from $\pi(\infty)$. This can always be done since $(1,1)$ or $(1,2) \notin \pi$ so as in $\pi(\infty)$ and the only common element is ∞.

- **Step 10:** Next list the $\{\infty, \infty_1, (1,3)\}$ block. This can be done because $(1,3)$ never occurs in $\pi(\infty)$.

- **Step 11:** Now list the blocks from B_3. Since the first block of B_3 is $\{(1,3),(2,3),(1\circ 2,1)\}$ the only common element is $(1,3)$.

- **Step 12:** Then swap the two blocks $\{(1,3),(3,3),(1\circ 3,1)\}$ and $\{(2,3),(3,3),(2\circ 3,1)\}$ in B_3. This can always be done since $(\alpha(2\circ 3),1)=(\alpha(1\circ 4),1)$ and $(\alpha(1\circ 2),1)\neq(\alpha(2\circ 3),1)$.

- **Step 13:** Finaly, write down the $\{(1,1),(1,2),(1,3)\}$ block. This can be done since $\alpha(1)=1$ we have $(\alpha(1\circ 3),1)\neq(1,1)$.

Then we will have a circular double change covering design of order v.

Note that in this construction we have

- 6 blocks in $F(\infty)$
- $\frac{v-6}{2}=3n$ blocks in $\pi(\infty)$
- $6n$ blocks of Type 2, since there are 6 blocks for each i and $1\leq i\leq n$
- $n(2n+1)*3$ blocks of Type 3.

Thus, the number of blocks $b=6n^2+12n+6$.

And

$$\frac{v(v-1)}{6}=\frac{(6n+6)(6n+5)}{6}=6n^2+11n+5.$$

Since $6n^2+12n+6>6n^2+11n+5$, this design is neither *economical* nor *tight* by the **Theorem 1.2.2**.

Example 3.3.2. Let $v = 12$ then $n = 1$ and the idempotent commutative quasigroup is as same as in Example 3.2.2. So we have

$X \setminus \{\infty_1, \infty_2, (1,1), (1,2), (1,3)\} = \{(2{,}1),(2{,}2),(2{,}3),(3{,}1),(3{,}2),(3{,}3)\}.$

Let $\pi = \{\{(2,1),(2,2)\},\{(2,3),(3,1)\},\{(3,2),(3,3)\}\}$ and then

$\pi(\infty) = \{\{\infty,(2,1),(2,2)\},\{\infty,(2,3),(3,1)\},\{\infty,(3,2),(3,3)\}\}$

Then following is a circular $dccd(12,3)$ with 24 blocks.

$\infty_1, \infty_2, (1,1)$	$(1,2),(2,2),(3,3)$	$\infty,(2,1),(2,2)$
$(1,1),(2,1),(3,2)$	$(1,2),(3,2),(2,3)$	$\infty,(2,3),(3,1)$
$(1,1),(3,1),(2,2)$	$(2,2),(3,2),(1,3)$	$\infty,(3,2),(3,3)$
$(2,1),(3,1),(1,2)$	$\infty_2,(2,2),(2,3)$	$\infty,\infty_1,(1,3)$
$\infty_1,(2,1),(2,2)$	$\infty_2,(3,1),(3,2)$	$(1,3),(2,3),(2,1)$
$\infty_1,(2,3),(3,1)$	$\infty_2,(2,1),(3,3)$	$(2,3),(3,3),(1,1)$
$\infty_1,(3,2),(3,3)$	$\infty,\infty_2,(1,3)$	$(1,3),(3,3),(3,1)$
$\infty_1,\infty_2,(1,2)$	$\infty,(1,1),(1,2)$	$(1,1),(1,2),(1,3)$

Example 3.3.3. Let $v = 18$ then $n = 2$ and use the same idempotent commutative quasi-group of order 5 as in Example 3.2.3. So we have

$$X \setminus \{\infty_1, \infty_2, (1,1), (1,2), (1,3)\} = \{(2,1),(2,2),(2,3),(3,1),(3,2),(3,3),$$
$$(4,1),(4,2),(4,3),(5,1),(5,2),(5,3)\}.$$

Let $\pi = \{\{(2,1),(2,2)\},\{(2,3),(3,1)\},\{(3,2),(3,3)\},\{(4,1),(4,2)\},\{(4,3),(5,1)\},\{(5,2),(5,3)\}\}$ and then

$$\pi(\infty) = \{\{\infty, (2,1), (2,2)\}, \{\infty, (2,3), (3,1)\}, \{\infty, (3,2), (3,3)\},$$
$$\{\infty, (4,1), (4,2)\}, \{\infty, (4,3), (5,1)\}, \{\infty, (5,2), (5,3)\}\}.$$

Then following is a circular $dccd(18, 3)$ with 54 blocks.

$\infty_1, \infty_2, (1,1)$	$(1,2), (2,2), (4,3)$	$\infty, (2,1), (2,2)$
$(1,1), (2,1), (4,2)$	$(1,2), (3,2), (2,3)$	$\infty, (2,3), (3,1)$
$(1,1), (3,1), (2,2)$	$(1,2), (4,2), (5,3)$	$\infty, (3,2), (3,3)$
$(1,1), (4,1), (5,2)$	$(1,2), (5,2), (3,3)$	$\infty, (4,1), (4,2)$
$(1,1), (5,1), (3,2)$	$(2,2), (5,2), (1,3)$	$\infty, (4,3), (5,1)$
$(2,1), (5,1), (1,2)$	$(3,2), (5,2), (4,3)$	$\infty, (5,2), (5,3)$
$(3,1), (5,1), (4,2)$	$(4,2), (5,2), (2,3)$	$\infty, \infty_1, (1,3)$
$(4,1), (5,1), (2,2)$	$(2,2), (4,2), (3,3)$	$(1,3), (2,3), (5,1)$
$(2,1), (4,1), (3,2)$	$(3,2), (4,2), (1,3)$	$(2,3), (3,3), (2,1)$
$(3,1), (4,1), (1,2)$	$(2,2), (3,2), (5,3)$	$(1,3), (4,3), (2,1)$
$(2,1), (3,1), (5,2)$	$\infty_2, (2,2), (2,3)$	$(1,3), (5,3), (4,1)$
$\infty_1, (2,1), (2,2)$	$\infty_2, (3,1), (3,2)$	$(2,3), (5,3), (1,1)$
$\infty_1, (2,3), (3,1)$	$\infty_2, (2,1), (3,3)$	$(3,3), (5,3), (5,1)$
$\infty_1, (3,2), (3,3)$	$\infty_2, (4,2), (4,3)$	$(4,3), (5,3), (3,1)$
$\infty_1, (4,1), (4,2)$	$\infty_2, (5,1), (5,2)$	$(2,3), (4,3), (4,1)$
$\infty_1, (4,3), (5,1)$	$\infty_2, (4,1), (3,3)$	$(3,3), (4,3), (1,1)$
$\infty_1, (5,2), (5,3)$	$\infty, \infty_2, (1,3)$	$(1,3), (3,3), (3,1)$
$\infty_1, \infty_2, (1,2)$	$\infty, (1,1), (1,2)$	$(1,1), (1,2), (1,3)$

3.4 CONSTRUCTION OF A CIRCULAR $DCCD$ FOR $V \equiv$ 2 OR 4 $\pmod 6$

The Tripole Covering Construction: Let $v \equiv 2$ or $4 \pmod 6$ and let (X, T) be a STS of order $v-1 \equiv 1$ or $3 \pmod 6$. Let $\{a, b, c\} \in T$ and let $\pi = \{\{x_1, y_1\}, \{x_2, y_2\}, \cdots, \{x_t, y_t\}\}$ be any partition of $X \setminus \{a, b, c\}$. Let $\pi(\infty) = \{\{\infty, x_1, y_1\}, \{\infty, x_2, y_2\}, \cdots, \{\infty, x_t, y_t\}\}$ and $T(\infty) = \{\{\infty, a, b\}, \{\infty, b, c\}, \{a, b, c\}\}$. Let $S = \{\infty\} \cup X$ then (S, T^*, P) is a minimum covering of order v, where $T^* = T \cup \pi(\infty) \cup T(\infty)$, and $P = \pi \cup \{\{a, b\}, \{b, c\}, \{\infty, b\}\}$.

Example 3.4.1. Let (X, T) be the Steiner Triple System in **Example 2.3.1** where $X = \{1, 2, 3, 4, 5, 6, 7\}$ and $T = \{\{1, 2, 3\}, \{2, 4, 7\}, \{3, 5, 7\}, \{1, 6, 7\}, \{1, 4, 5\}, \{2, 5, 6\}, \{3, 4, 6\}\}$. Let $\pi = \{\{4, 5\}, \{6, 7\}\}$ be a partition of $X \setminus \{1, 2, 3\}$ then

$\pi(\infty) = \{\{\infty, 4, 5\}, \{\infty, 6, 7\}\}$ and

$T(\infty) = \{\{\infty, 1, 2\}, \{\infty, 2, 3\}, \{1, 2, 3\}\}$.

Then (S,T^*,P) is a minimum covering of order 8, where $T^* = T \cup \pi(\infty) \cup T(\infty)$ and $P = \pi \cup \{\{1, 2\}, \{2, 3\}, \{\infty, 2\}\}$

Example 3.4.2. Let (X, T) be the Steiner Triple System in **Example 2.2.1** where $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Let $\pi = \{\{4, 5\}, \{6, 7\}, \{8, 9\}\}$ be a partition of $X \setminus \{1, 2, 3\}$ then

$\pi(\infty) = \{\{\infty, 4, 5\}, \{\infty, 6, 7\}, \{\infty, 8, 9\}$ and

$T(\infty) = \{\{\infty, 1, 2\}, \{\infty, 2, 3\}, \{1, 2, 3\}$.

Then (S,T^*,P) is a minimum covering of order 10, where $T^* = T \cup \pi(\infty) \cup T(\infty)$ and $P = \pi \cup \{\{1, 2\}, \{2, 3\}, \{\infty, 2\}\}$

3.4.1 Construction of a circular $dccd$ for $v \equiv 4 \pmod 6$

Consider the $6n + 3$ construction with the Type 1 block $\{(1, 1), (1, 2), (1, 3)\} \in T$. Then $T(\infty)$ contains the following three blocks:

$\{\infty, (1, 1), (1, 2)\}$

$\{\infty, (1, 2), (1, 3)\}$

$\{(1, 1), (1, 2), (1, 3)\}$

Let $\pi = \{\{(2i+2,1),(2i+3,1)\},\{(2i+2,2),(2i+3,2)\},\{(2i+2,3),(2i+3,3)\}\}$ where $0 \leq i \leq n-1$, be a partition of $X \setminus \{(1,1),(1,2),(1,3)\}$.

Since π partitions the set $X \setminus \{(1,1),(1,2),(1,3)\}$, blocks of $\pi(\infty)$ form a double change design with ∞ as the common element. Then the following is a procedure to construct a circular double change covering design for $n \geq 1$.

- **Step 1:** List down the circular *dccd* for $6n+3$.
- **Step 2:** Insert the blocks of $\pi(\infty)$ after the block $\{(2,1),(2,2),(2,3)\}$ (i.e. after Step 2 in the $6n+3$ construction). This can be always done since the first block of $\pi(\infty)$ is $\{\infty,(2,1),(3,1)\}$ and (2,1) is the common element.
- **Step 3:** Insert the block $\{\infty,(1,2),(1,3)\}$ from $T(\infty)$ in between any two blocks of $\pi(\infty)$. This can be done since (1,2) or (1,3) $\notin \pi$ so the only common element is ∞ and π contains more than 2 elements.
- **Step 4:** Insert the block $\{\infty,(1,1),(1,2)\}$ from $T(\infty)$. This can be done since (1,1) or (1,2) $\notin \pi$ so the only common element is ∞.
- **Step 5:** Continue from Step 3 in the $6n+3$ construction. This can always be done since the first block of B_2 is $\{(1,2),(2,2),(1 \circ 2,3)\}$ and only (1,2) is common.

Note that in this construction we have

- $6n^2+5n+1$ blocks in T.
- 2 blocks in $T(\infty)$. Since the block $\{(1,1),(1,2),(1,3)\} \in T$ there are only 2 new blocks in $T(\infty)$.
- $3n$ blocks in $\pi(\infty)$.

Thus, the number of blocks $b = 6n^2 + 8n + 3$. And

$$\frac{v(v-1)}{6} = \frac{(6n+4)(6n+3)}{6} = 6n^2 + 7n + 2.$$

Since $\lceil 6n^2 + 7n + 2 \rceil < 6n^2 + 8n + 3$, this design is neither *economical* nor *tight* by the **Theorem 1.2.2**.

Example 3.4.3. Let $v = 10$ then $n = 1$ and we will use a $dccd(9,3)$ to construct a $dccd(10,3)$ with 17 blocks. Here we have

$T(\infty) = \{\{\infty, (1,1), (1,2)\}, \{\infty, (1,2), (1,3)\}, \{(1,1), (1,2), (1,3)\}\}$

$\pi = \{\{(2,1), (3,1)\}, \{(2,2), (3,2)\}, \{(2,3), (3,3)\}\}$

$\pi(\infty) = \{\{\infty, (2,1), (3,1)\}, \{\infty, (2,2), (3,2)\}, \{\infty, (2,3), (3,3)\}\}$

Then the following is a circular $dccd(10,3)$ with 17 blocks.

$(1,1), (2,1), (3,2)$	$\infty, (2,2), (3,2)$	$(1,1), (1,2), (1,3)$
$(1,1), (3,1), (2,2)$	$\infty, (2,3), (3,3)$	$(1,3), (2,3), (3,1)$
$(2,1), (3,1), (1,2)$	$\infty, (1,1), (1,2)$	$(1,3), (3,3), (2,1)$
$(2,1), (2,2), (2,3)$	$(1,2), (2,2), (3,3)$	$(2,3), (3,3), (1,1)$
$\infty, (2,1), (3,1)$	$(1,2), (3,2), (2,3)$	$(3,1), (3,2), (3,3)$
$\infty, (1,2), (1,3)$	$(2,2), (3,2), (1,3)$	

Example 3.4.4. Let $v = 16$ then $n = 2$ and the following is a circular $dccd(16, 3)$ with 43 blocks. Here we have

$T(\infty) = \{\{\infty, (1,1), (1,2)\}, \{\infty, (1,2), (1,3)\}, \{(1,1), (1,2), (1,3)\}\}$

$\pi = \{\{(2,1), (3,1)\}, \{(2,2), (3,2)\}, \{(2,3), (3,3)\},$
$\{(4,1), (5,1)\}, \{(4,2), (5,2)\}, \{(4,3), (5,3)\}\}$

$\pi(\infty) = \{\{\infty, (2,1), (3,1)\}, \{\infty, (2,2), (3,2)\}, \{\infty, (2,3), (3,3)\},$
$\{\infty, (4,1), (5,1)\}, \{\infty, (4,2), (5,2)\}, \{\infty, (4,3), (5,3)\}\}$

and the $dccd(16, 3)$ is:

$(1,1),(3,1),(2,2)$	$\infty,(2,2),(3,2)$	$(3,2),(4,2),(1,3)$
$(1,1),(2,1),(4,2)$	$\infty,(2,3),(3,3)$	$(1,2),(3,2),(2,3)$
$(1,1),(4,1),(5,2)$	$\infty,(4,1),(5,1)$	$(1,1),(1,2),(1,3)$
$(1,1),(5,1),(3,2)$	$\infty,(4,2),(5,2)$	$(1,3),(2,3),(4,1)$
$(5,1),(5,2),(5,3)$	$\infty,(4,3),(5,3)$	$(1,3),(3,3),(2,1)$
$(2,1),(5,1),(1,2)$	$\infty,(1,1),(1,2)$	$(1,3),(4,3),(5,1)$
$(3,1),(5,1),(4,2)$	$(1,2),(2,2),(4,3)$	$(1,3),(5,3),(3,1)$
$(4,1),(5,1),(2,2)$	$(2,2),(3,2),(5,3)$	$(2,3),(5,3),(1,1)$
$(4,1),(4,2),(4,3)$	$(1,2),(4,2),(5,3)$	$(3,3),(5,3),(4,1)$
$(2,1),(4,1),(3,2)$	$(1,2),(5,2),(3,3)$	$(4,3),(5,3),(2,1)$
$(3,1),(4,1),(1,2)$	$(2,2),(5,2),(1,3)$	$(2,3),(4,3),(3,1)$
$(2,1),(3,1),(5,2)$	$(3,2),(5,2),(4,3)$	$(3,3),(4,3),(1,1)$
$(2,1),(2,2),(2,3)$	$(4,2),(5,2),(2,3)$	$(2,3),(3,3),(5,1)$
$\infty,(2,1),(3,1)$	$(2,2),(4,2),(3,3)$	$(3,1),(3,2),(3,3)$
$\infty,(1,2),(1,3)$		

3.4.2 Construction of a circular *dccd* for $v \equiv 2 \pmod 6$

Let (X, T) be a STS of order $v - 1 \equiv 1 \pmod 6$ where $X = \{\infty_1\} \cup (Q \times \{1, 2, 3\})$. Consider the Type 1 block $\{(1,1),(1,2),(1,3)\} \in T$ then $T(\infty)$ contains the following three blocks:

$$\{\infty, (1,1), (1,2)\}$$
$$\{\infty, (1,2), (1,3)\}$$
$$\{(1,1), (1,2), (1,3)\}$$

Let $\pi = (\{(2i,1),(2i+1,1)\}, \{(2i,2),(2i+1,2)\}, \{(2i,3),(2i+1,3)\} \;\; ; \; 1 \leq i \leq n-1)$
$\cup(\{(2n,1),(2n,2)\}, \{\infty_1,(2n,3)\})$ be a partition of $X \setminus \{(1,1),(1,2),(1,3)\}$.

Example 3.4.5. Let $v = 8$ then $n = 1$ and consider the half-idempotent commutative quassigroup of order 2 in **Example 2.3.1** and let $X = \{\infty_1\} \cup (\{1, 2\} \times \{1, 2, 3\})$. Then we have

$$\pi = \{\{(2,1),(2,2)\}, \{\infty_1,(2,3)\}\} \text{ and then}$$
$$\pi(\infty) = \{\{\infty,(2,1),(2,2)\}, \{\infty, \infty_1,(2,3)\}\}$$

Then the following is a circular $dccd(8, 3)$ with 11 blocks. i.e. the $6n+2$ construction when $n = 1$.

(1,1)	(1,2)	(1,3)
∞_1	(2,1)	(1,2)
∞_1	(2,2)	(1,3)
∞_1	(2,3)	(1,1)
(1,1)	(2,1)	(2,2)
(1,2)	(2,2)	(2,3)
∞	(1,1)	(1,2)
∞	(2,1)	(2,2)
∞	∞_1	(2,3)
∞	(1,2)	(1,3)
(1,3)	(2,3)	(2,1)

Since π partitions the set $X\backslash\{(1,1),(1,2),(1,3)\}$, blocks of $\pi(\infty)$ form a double change design with ∞ as the common element. Then the following is a procedure to construct a circular double change covering design for $n \geq 2$.

- **Step 1:** List the circular *dccd* for $6n+1$.
- **Step 2:** Insert the blocks of $\pi(\infty)$ after the block $\{(2,1),(2,2),(2,3)\}$ (i.e. after Step 4 in the $6n+1$ construction). This can be always done since the first block of $\pi(\infty)$ is $\{\infty,(2,1),(3,1)\}$ and (2,1) is the common element.
- **Step 3:** Insert the block $\{\infty,(1,2),(1,3)\}$ from $T(\infty)$ in between any two blocks of $\pi(\infty)$. This can be done since (1,2) or (1,3) $\notin \pi$ so the only common element is ∞ and π contains more than 2 elements.

- **Step 4:** Insert the block $\{\infty, (1,1), (1,2)\}$ from $T(\infty)$. This can be done since (1,1) or (1,2) $\notin \pi$ so the only common element is ∞.

- **Step 5:** Continue from Step 5 in the $6n+1$ construction. This can always be done since the first block of B_2 is $\{(1,2), (2,2), (1 \circ 2, 3)\}$ and only (1,2) is common.

Note that in this construction we have

- $6n^2 + n$ blocks in T.

- 2 blocks in $T(\infty)$. Since the block $\{(1,1), (1,2), (1,3)\} \in T$ there are only 2 new blocks in $T(\infty)$.

- $3n - 1$ blocks in $\pi(\infty)$.

Thus, the number of blocks $b = 6n^2 + 4n + 1$. And

$$\frac{v(v-1)}{6} = \frac{(6n+2)(6n+1)}{6} = \frac{18n^2 + 9n + 1}{3}.$$

Since $\left\lceil \frac{18n^2+9n+1}{3} \right\rceil = 6n^2 + 3n + 1 < 6n^2 + 4n + 1$, this design is neither *economical* nor *tight* by the **Theorem 1.2.2**.

Example 3.4.6. Let $v = 14$ then $n = 2$ and we have

$\pi = \{\{(2,1), (3,1)\}, \{(2,2), (3,2)\}, \{(2,3), (3,3)\}, \{(4,1), (4,2)\}, \{\infty_1, (4,3)\}\}$ and

$\pi(\infty) = \{\{\infty, (2,1), (3,1)\}, \{\infty, (2,2), (3,2)\}, \{\infty, (2,3), (3,3)\},$

$\{\infty, (4,1), (4,2)\}, \{\infty, \infty_1, (4,3)\}\}$

Then following is a circular $dccd(14, 3)$ with 33 blocks.

$(1,1),(1,2),(1,3)$	$(3,1),(4,1),(3,2)$	$(1,2),(3,2),(2,3)$
$\infty_1,(3,1),(1,2)$	$(2,1),(3,1),(4,2)$	$(1,2),(4,2),(4,3)$
$\infty_1,(4,1),(2,2)$	$(2,1),(2,2),(2,3)$	$(2,2),(4,2),(1,3)$
$\infty_1,(3,2),(1,3)$	$\infty,(2,1),(3,1)$	$(1,3),(2,3),(3,1)$
$\infty_1,(4,2),(2,3)$	$\infty,(1,2),(1,3)$	$(1,3),(3,3),(2,1)$
$\infty_1,(3,3),(1,1)$	$\infty,(2,2),(3,2)$	$(3,2),(4,2),(3,3)$
$\infty_1,(4,3),(2,1)$	$\infty,(2,3),(3,3)$	$(2,2),(3,2),(4,3)$
$(1,1),(2,1),(3,2)$	$\infty,(4,1),(4,2)$	$(2,3),(4,3),(1,1)$
$(1,1),(3,1),(2,2)$	$\infty,\infty_1,(4,3)$	$(3,3),(4,3),(3,1)$
$(1,1),(4,1),(4,2)$	$\infty,(1,1),(1,2)$	$(2,3),(3,3),(4,1)$
$(2,1),(4,1),(1,2)$	$(1,2),(2,2),(3,3)$	$(1,3),(4,3),(4,1)$

Example 3.4.7. Let $v = 20$ then $n = 3$ and we have

$$\begin{aligned}\pi &= \{\{(2,1),(3,1)\},\{(2,2),(3,2)\},\{(2,3),(3,3)\},\{(4,1),(5,1)\},\\ &\quad \{(4,2),(5,2)\},\{(4,3),(5,3)\},\{(6,1),(6,2)\},\{\infty_1,(6,3)\}\} \text{ and}\\ \pi(\infty) &= \{\{\infty,(2,1),(3,1)\},\{\infty,(2,2),(3,2)\},\{\infty,(2,3),(3,3)\},\{\infty,(4,1),(5,1)\},\\ &\quad \{\infty,(4,2),(5,2)\},\{\infty,(4,3),(5,3)\},\{\infty,(6,1),(6,2)\},\{\infty,\infty_1,(6,3)\}\}\end{aligned}$$

Then following is a circular $dccd(19,3)$ with 67 blocks.

$(1,1),(1,2),(1,3)$	$(3,1),(4,1),(6,2)$	$(2,2),(5,2),(6,3)$
$\infty_1,(4,1),(1,2)$	$(2,1),(3,1),(5,2)$	$(3,2),(5,2),(1,3)$
$\infty_1,(5,1),(2,2)$	$(2,1),(2,2),(2,3)$	$(4,2),(5,2),(4,3)$
$\infty_1,(6,1),(3,2)$	$\infty,(2,1),(3,1)$	$(2,2),(4,2),(3,3)$
$\infty_1,(4,2),(1,3)$	$\infty,(1,2),(1,3)$	$(3,2),(4,2),(6,3)$
$\infty_1,(5,2),(2,3)$	$\infty,(2,2),(3,2)$	$(2,2),(3,2),(5,3)$
$\infty_1,(6,2),(3,3)$	$\infty,(2,3),(3,3)$	$(3,1),(3,2),(3,3)$
$\infty_1,(4,3),(1,1)$	$\infty,(4,1),(5,1)$	$(1,3),(3,3),(2,1)$
$\infty_1,(5,3),(2,1)$	$\infty,(4,2),(5,2)$	$(1,3),(4,3),(5,1)$
$\infty_1,(6,3),(3,1)$	$\infty,(4,3),(5,3)$	$(1,3),(5,3),(3,1)$
$(1,1),(3,1),(2,2)$	$\infty,(6,1),(6,2)$	$(1,3),(6,3),(6,1)$
$(1,1),(2,1),(4,2)$	$\infty,\infty_1,(6,3)$	$(2,3),(6,3),(1,1)$
$(1,1),(4,1),(5,2)$	$\infty,(1,1),(1,2)$	$(3,3),(6,3),(4,1)$
$(1,1),(5,1),(3,2)$	$(1,2),(2,2),(4,3)$	$(4,3),(6,3),(2,1)$
$(1,1),(6,1),(6,2)$	$(1,2),(3,2),(2,3)$	$(5,3),(6,3),(5,1)$
$(2,1),(6,1),(1,2)$	$(1,2),(4,2),(5,3)$	$(2,3),(5,3),(6,1)$
$(3,1),(6,1),(4,2)$	$(1,2),(5,2),(3,3)$	$(3,3),(5,3),(1,1)$
$(4,1),(6,1),(2,2)$	$(1,2),(6,2),(6,3)$	$(4,3),(5,3),(4,1)$
$(5,1),(6,1),(5,2)$	$(2,2),(6,2),(1,3)$	$(2,3),(4,3),(3,1)$
$(2,1),(5,1),(6,2)$	$(3,2),(6,2),(4,3)$	$(3,3),(4,3),(6,1)$
$(3,1),(5,1),(1,2)$	$(4,2),(6,2),(2,3)$	$(2,3),(3,3),(5,1)$
$(4,1),(5,1),(4,2)$	$(5,2),(6,2),(5,3)$	$(1,3),(2,3),(4,1)$
$(2,1),(4,1),(3,2)$		

3.5 CONCLUSION

To answer the question given in the first section of the paper about the car testing, suppose the company has 11 types of cars and each driver tests exactly 3 cars and drives only one type of cars as the previous driver. Since $k = 3$ we have number of drivers $b = 19$. So problem asks to construct a circular $dccd(11, 3)$ with 19 blocks. We see that $11 \equiv 5 \pmod 6$, so we use the $6n + 5$ construction and then arrange the blocks using the method described to get a circular double-change covering design. If we get the eleven cars be $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$, following is an *economical* circular $dccd(11, 3)$ with 19 blocks.

1	2	3	5	7	10
3	6	10	2	7	8
3	7	9	2	9	10
4	6	9	2	6	11
1	6	7	1	2	5
1	8	9	5	6	8
1	10	11	3	8	11
1	2	4	5	9	11
4	7	11	3	4	5
4	8	10			

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